

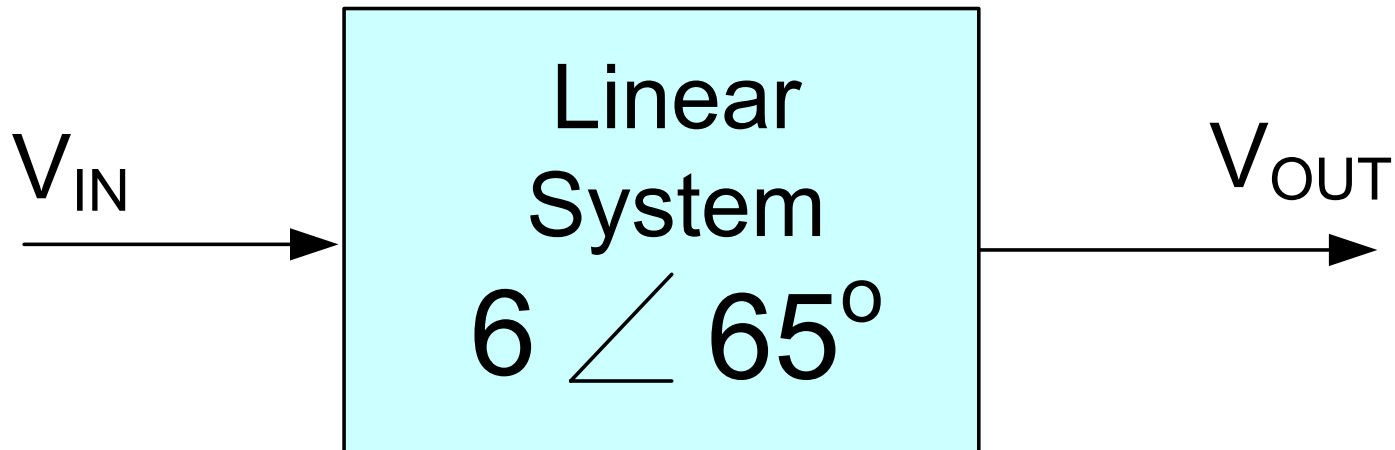
EE 230

Lecture 4

Transfer Characteristics and  
Transfer Functions

# Quiz 3

A linear system has a phasor domain transfer function of  $6 \angle 65^\circ$ . If the input to the system is  $V_{IN} = 5\sin(3000t + 25^\circ)$ , determine the steady state output voltage.



And the number is ?

1

3

8

5

4

2

6

9

7

And the number is ?

1

3

8

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2

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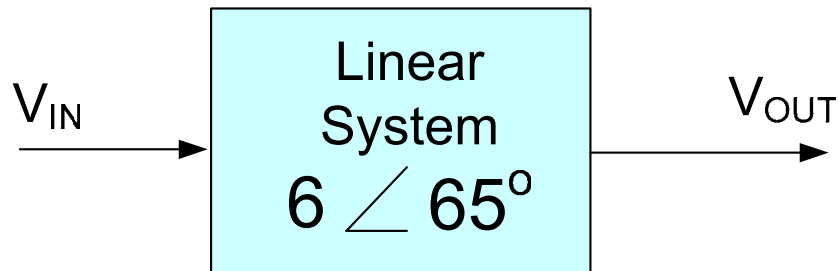
6

9

7

# Quiz 3

A linear system has a phasor domain transfer function of  $6 \angle 65^\circ$ . If the input to the system is  $V_{IN} = 5\sin(3000t + 25^\circ)$ , determine the steady state output voltage.



Solution:

$$V_{OSS} = V_M |T_P(j\omega)| \sin(\omega t + \theta + \gamma_{T_P(j\omega)})$$

$$V_{OSS} = 5 \cdot 6 \sin(3000t + 25^\circ + 65^\circ)$$

$$V_{OSS} = 30 \sin(3000t + 90^\circ)$$

# Review from Last Time

Distortion of major concern in many electronic systems

-THD is one metric that is often used to characterize amount of distortion

Amplifiers are circuits that scale the input to a linear system by a constant

Sinusoidal steady state response of linear system given by

$$V_{OSS} = V_M |T_P(j\omega)| \sin(\omega t + \theta + \gamma_{T_P(j\omega)})$$

where  $T_P(j\omega)$  is the phasor-domain transfer function

One of the most important concepts introduced in EE 201

Electronics and systems community seldom uses concept of phasor domain transfer function

Electronics and systems community critically dependent upon sinusoidal steady state response

What is used by the electronics and systems community ?

Why is  $T_P(j\omega)$  seldom used ?

- Authors of current electronics textbook do not talk about phasors or  $T_p(j\omega)$
- This is consistent with the industry when discussing electronic circuits and systems
- The sinusoidal steady state response is of considerable concern in electronic circuits and is used extensively in this text
- Authors & industry use concept of transfer function  $T(s)$

How does  $T(s)$  relate to  $T_p(j\omega)$  ?

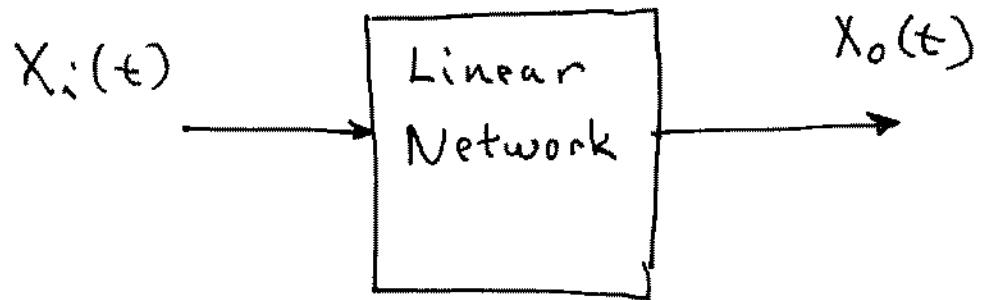
Why is  $T(s)$  used instead of  $T_p(j\omega)$  ?

What is  $T(s)$  ?

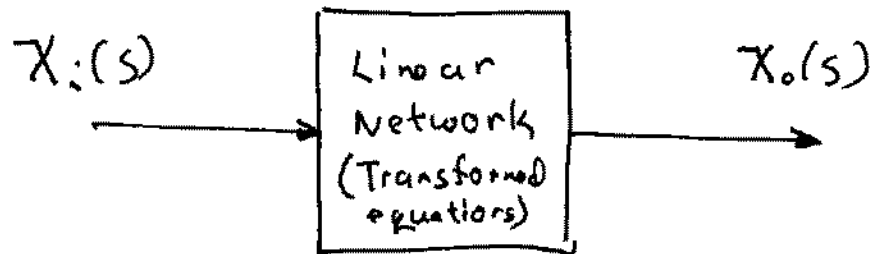
Why was  $T_p(j\omega)$  used in EE201 instead of  $T(s)$  for characterizing frequency dependence of linear networks ?



What is  $T(s)$ ?



using Laplace Transform



$$\frac{X_o(s)}{X_i(s)} = T(s)$$

What is  $T(s)$  ?

$T(s)$  is the ratio of the Laplace Transform of the output to the Laplace Transform of the input

$T(s)$  is called the transfer function

Theorem: If the input to a linear network with transfer function  $T(s)$  is  $V_m \sin(\omega t + \gamma)$ , then the sinusoidal steady state response is

$$V_o(t) = V_m |T(j\omega)| \sin(\omega t + \gamma + \angle T(j\omega))$$

In the differential equations class,  
 $T(s)$  was obtained by taking the  
Laplace transform of a set of differential  
equations of a linear system

Why was the Laplace Transform  
concept introduced in the differential  
equations class?

The Laplace Transform of a set of differential equations in the time domain resulted in a set of Linear equations in the transformed domain

The resultant set of linear equations was much easier to solve in most cases than the set of differential equations

$$X_o(t) = \int_{\infty}^{-1} T(s) X_i(s)$$

How does  $T(s)$  relate to  $T_p(j\omega)$  ?

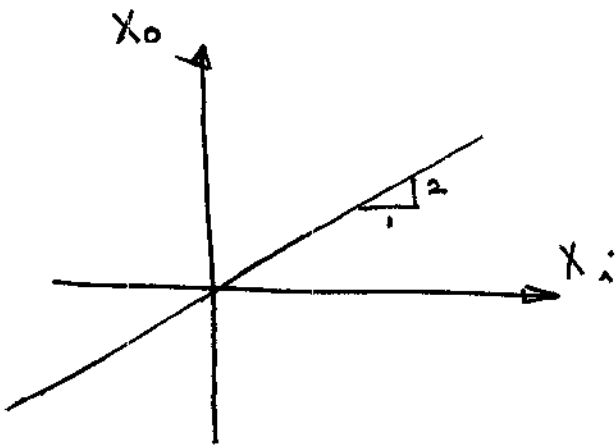
$$T(s) \Big|_{s=j\omega} = T_p(j\omega)$$

We know how to determine  $T_p(j\omega)$ . Is there an easy way to obtain  $T(s)$  ?

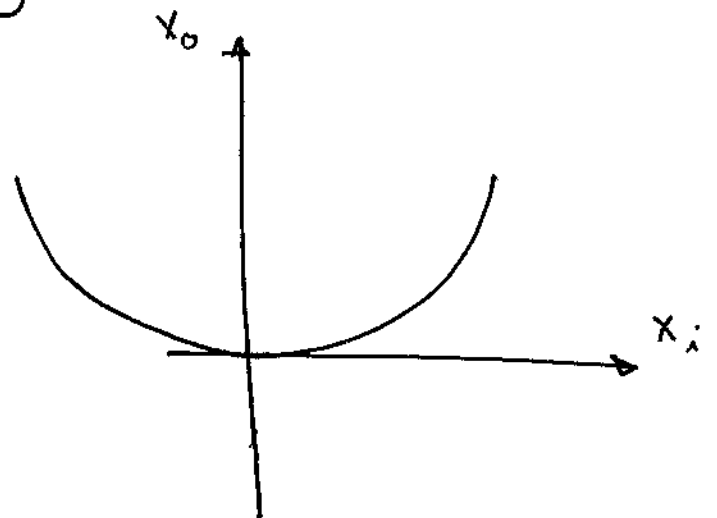
# Transfer Characteristic & Transfer Function

## Transfer Characteristics

Relationship between input and output variable (input variable is very slowly varying or not varying)



$$X_o = 2X_i$$

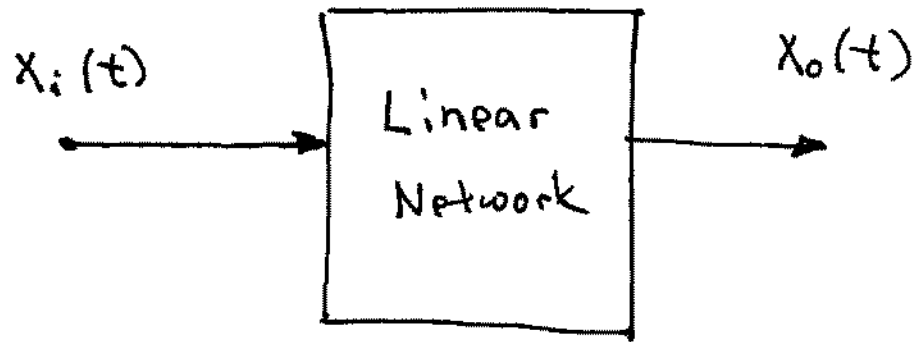


$$X_o = X_i^2$$

If the relationship between  $X_o$  &  $X_i$  is linear

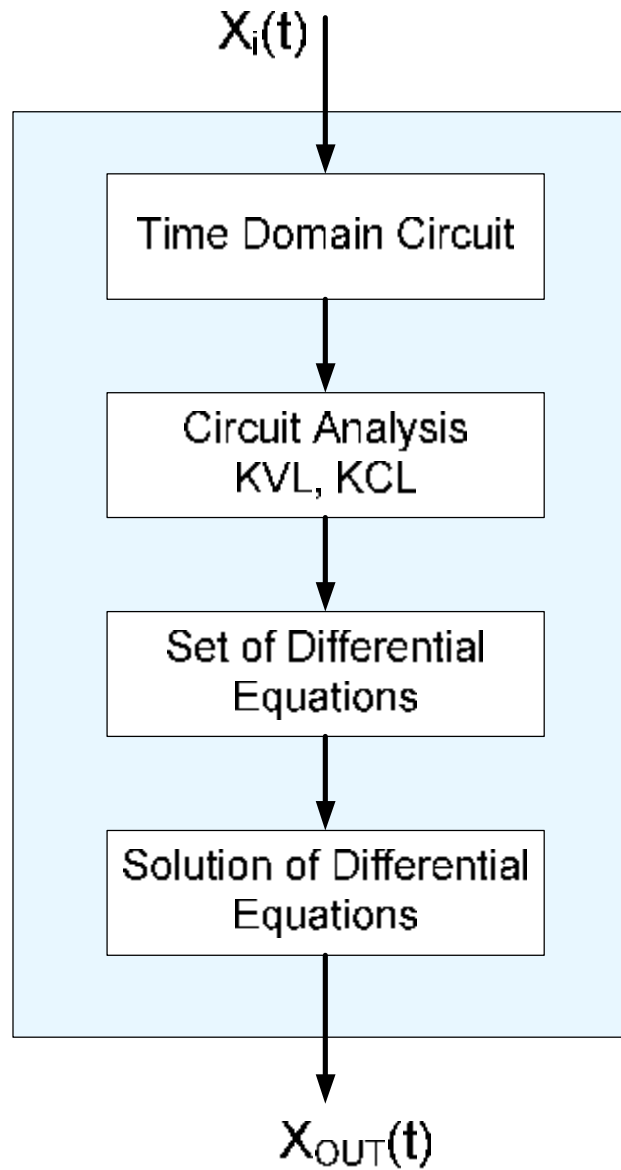


# Transfer Function

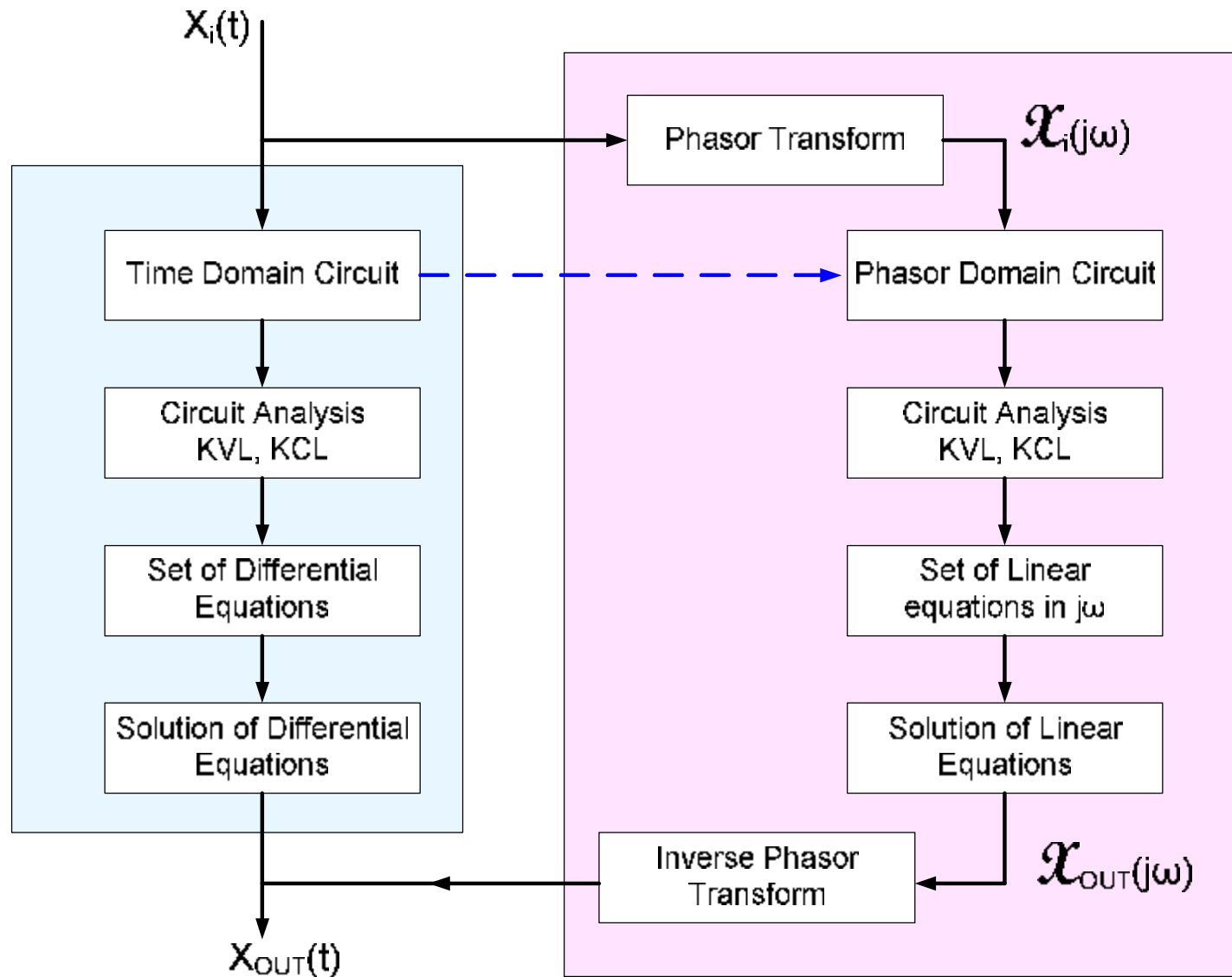


$$T_p(j\omega) = \frac{\vec{X}_o(j\omega)}{\vec{X}_i(j\omega)}$$

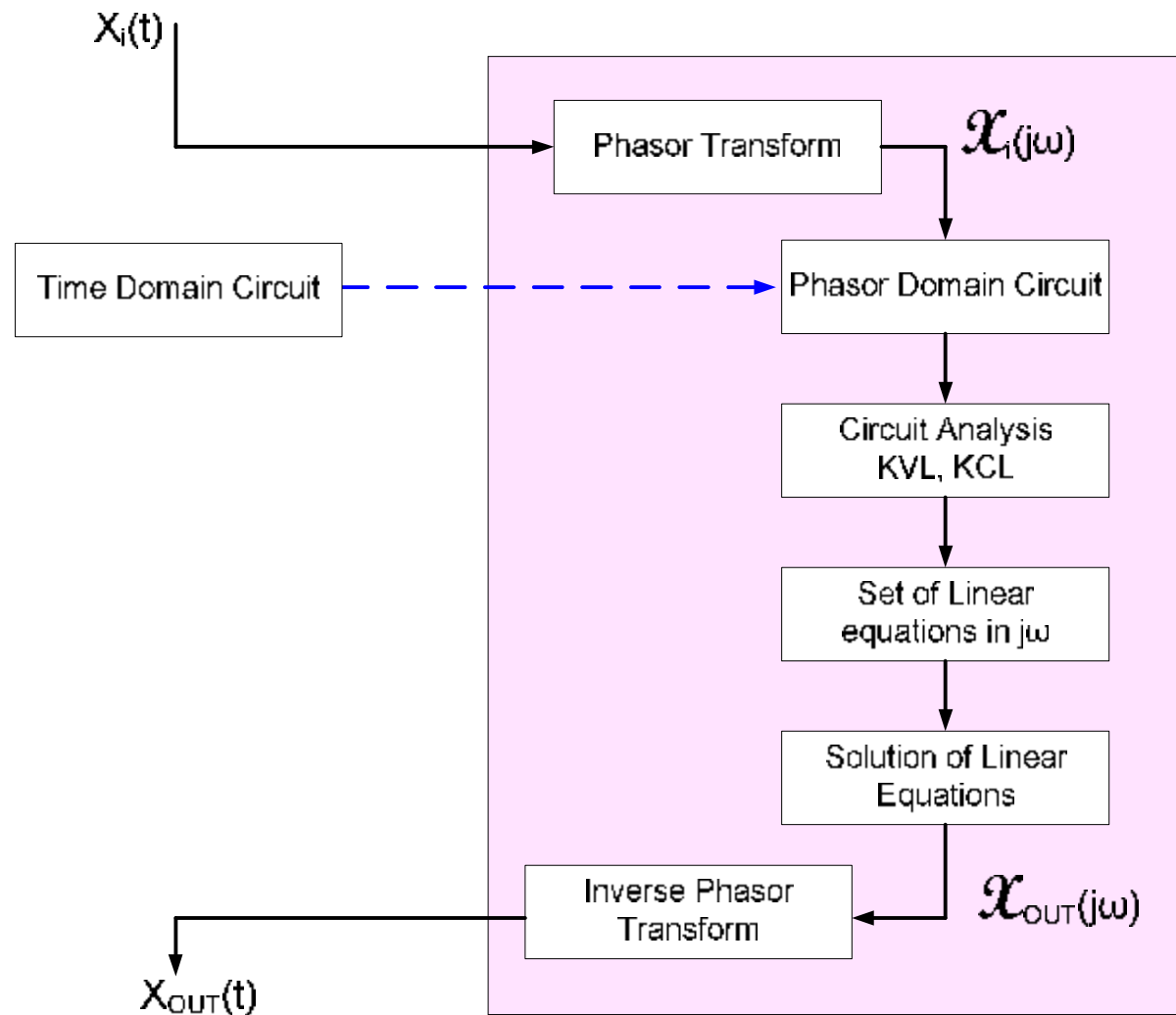
## Standard Approach to Circuit Analysis



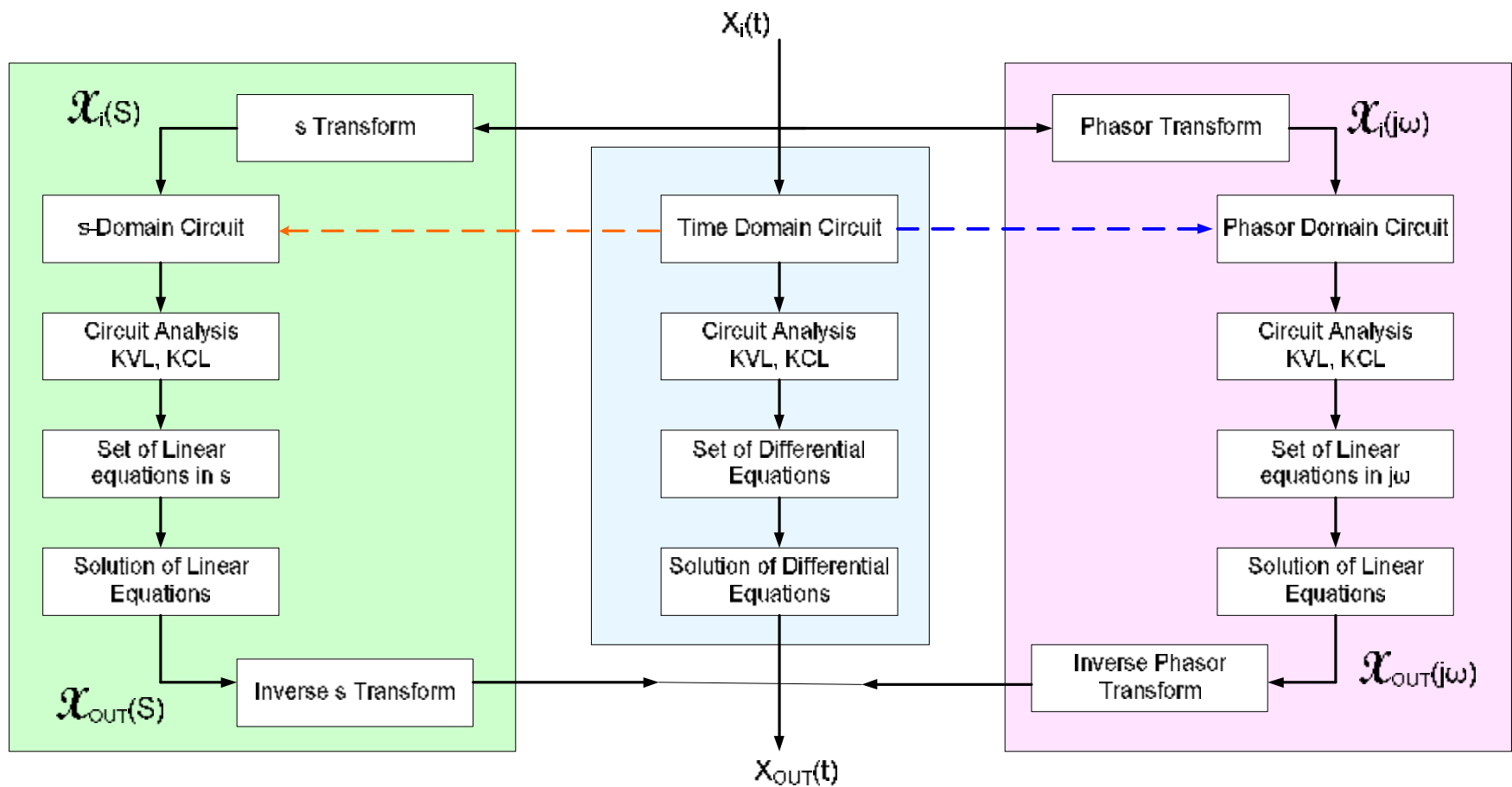
# Time and Phasor Domain Analysis



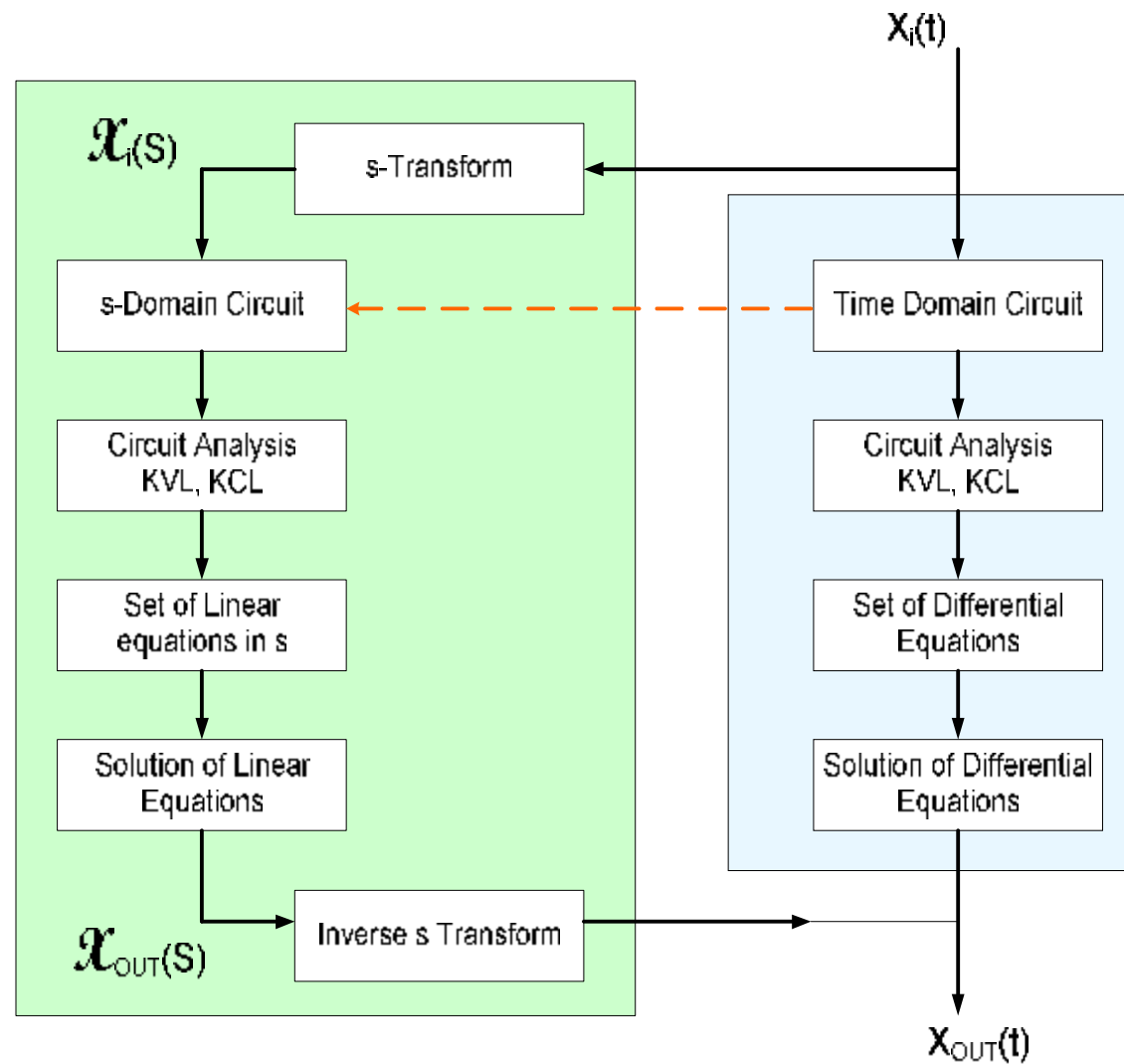
# Phasor Domain Analysis



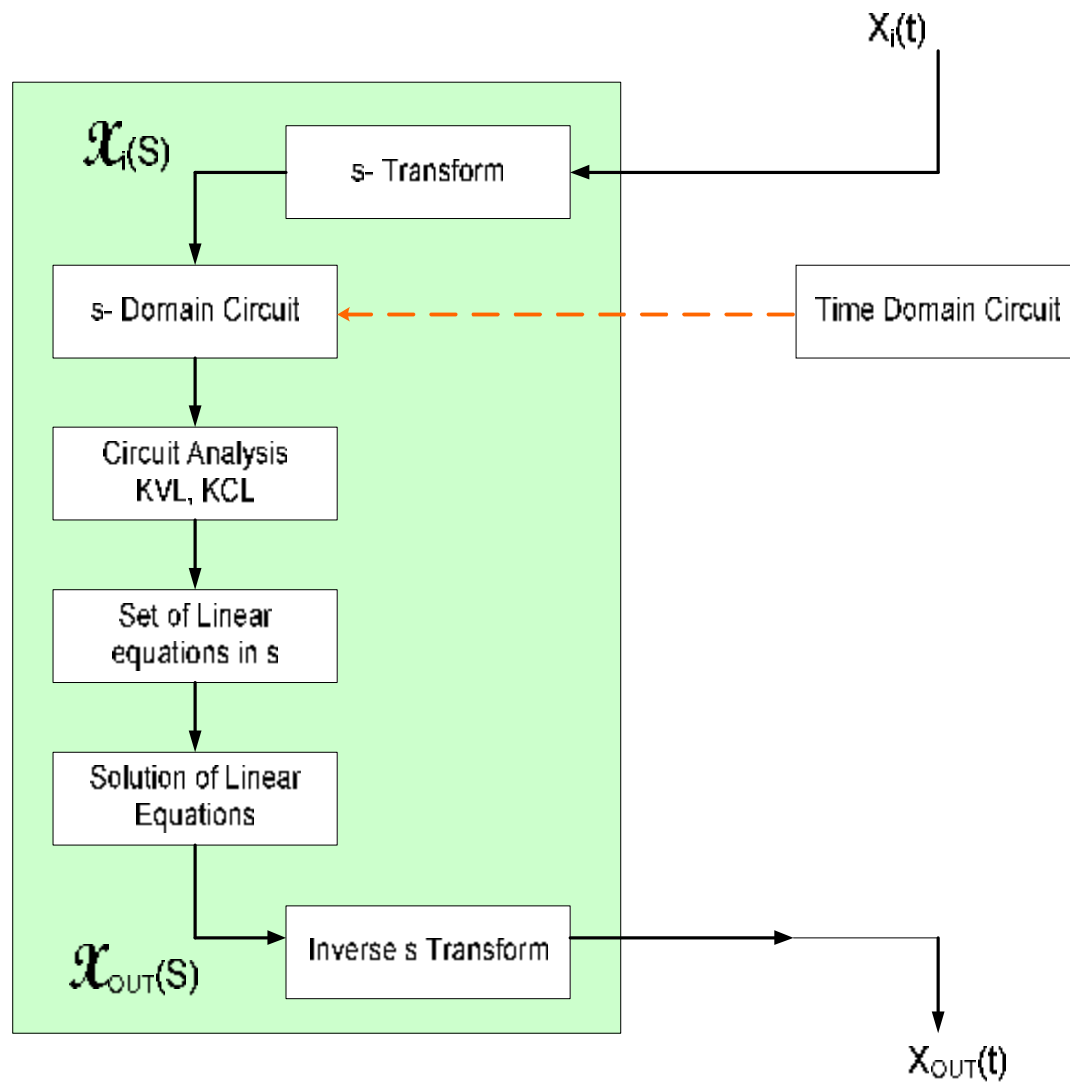
# Time, Phasor, and s- Domain Analysis



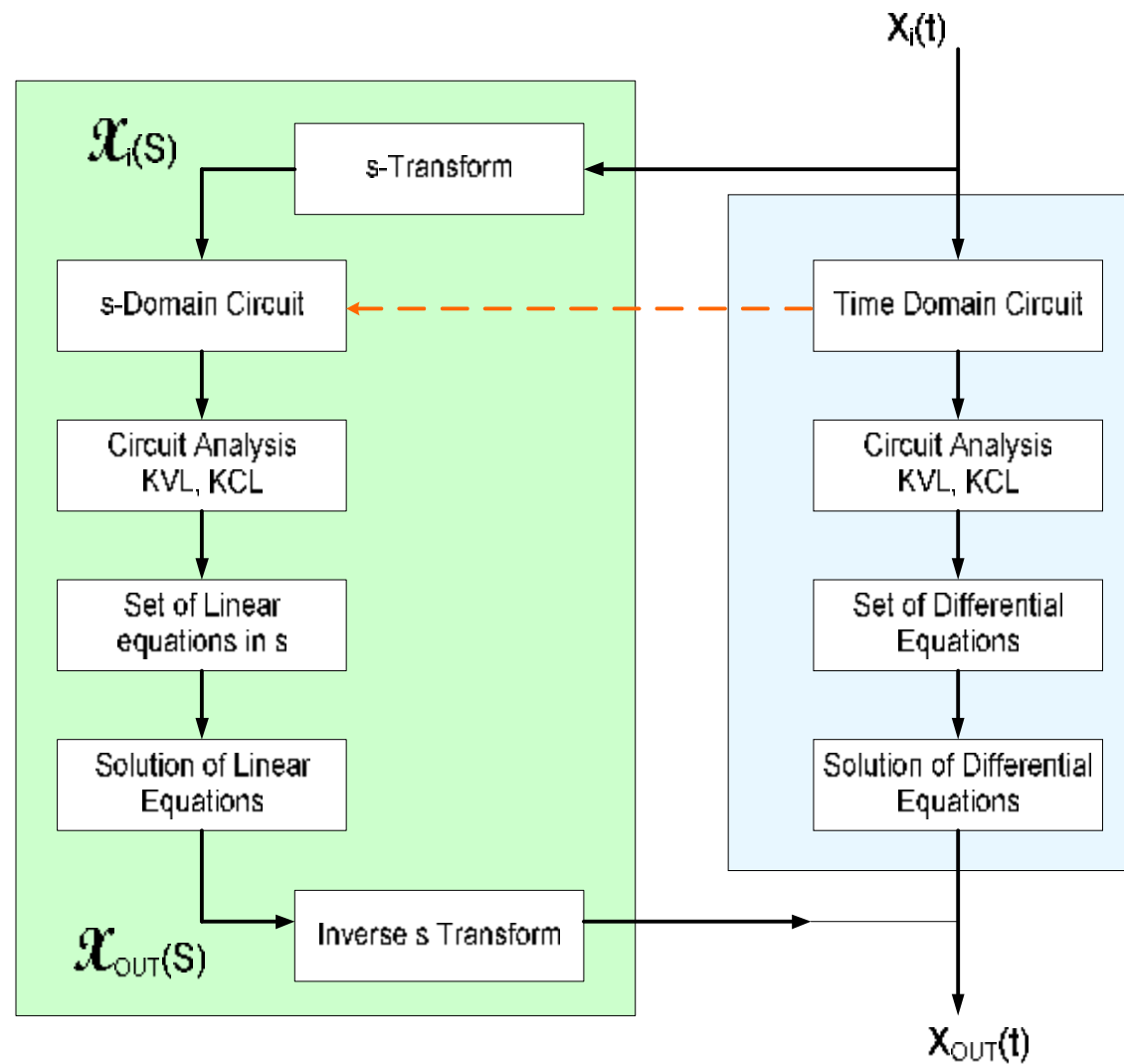
# Time and s- Domain Analysis



# s- Domain Analysis

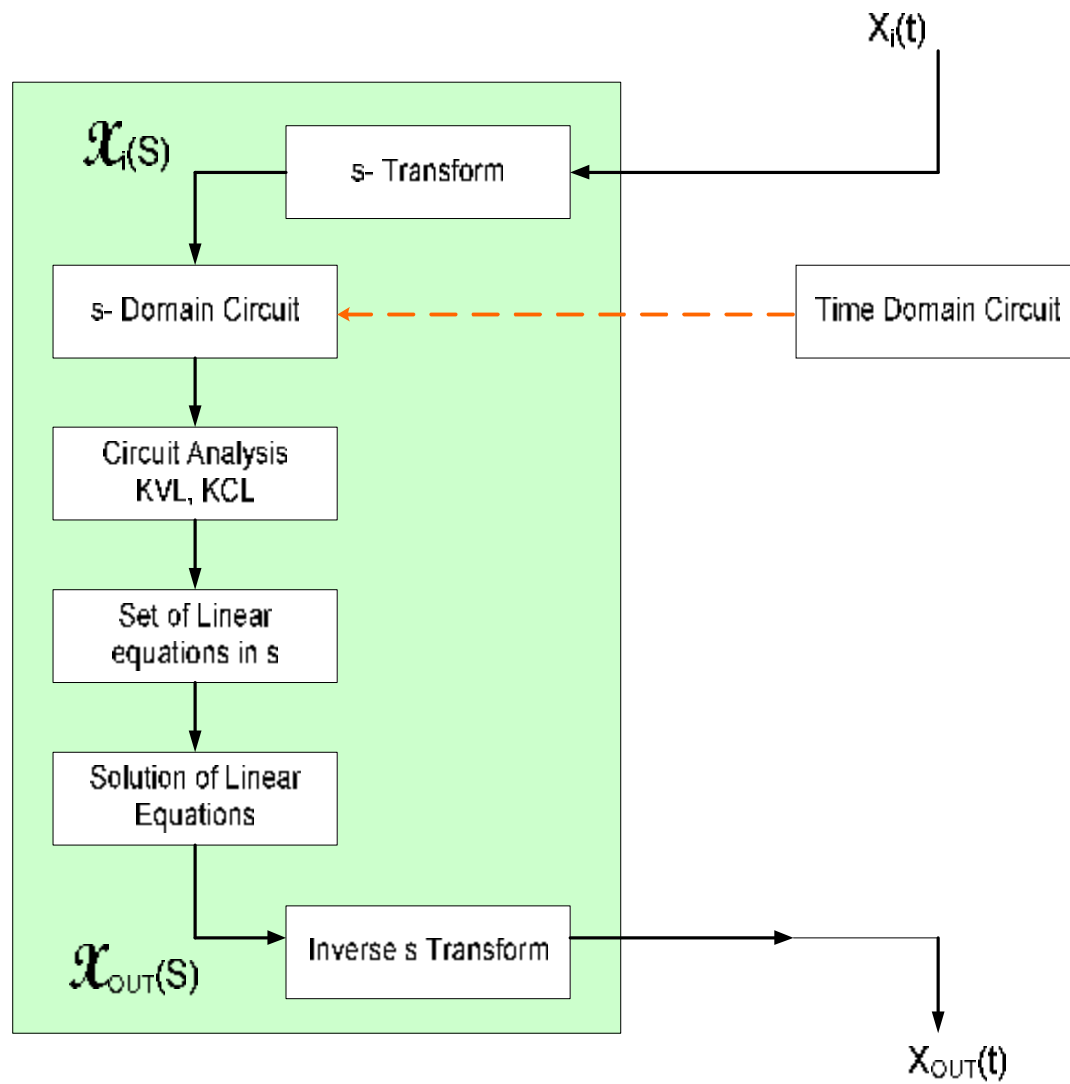


# Time and s- Domain Analysis





# s- Domain Analysis



will show today

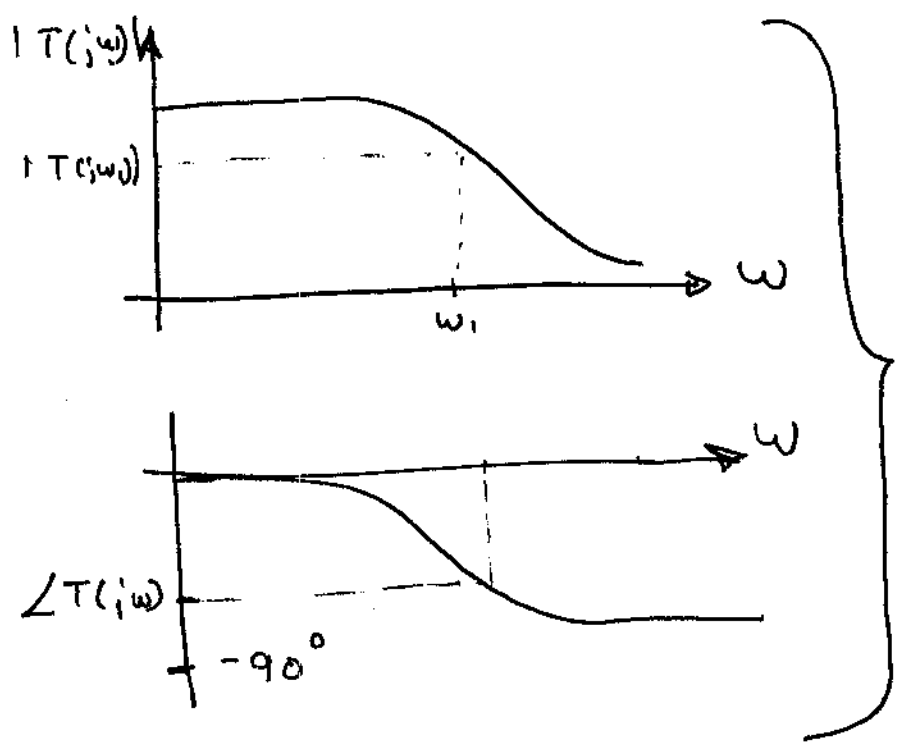
1)  $T(s)$  can often be obtained directly without even writing the differential equation

$$2) T(s) \Big|_{s=j\omega} = T_p(\omega)$$

3) If  $x_i = x_m \sin(\omega t + \theta)$

$$x_{oss} = x_m |T(j\omega)| \sin(\omega t + \theta + \angle T(j\omega))$$

If  $|T(j\omega)|$  and  $\angle T(j\omega)$  are plotted



magnitude & phase plots

$$V_o(t) = X_m |T(j\omega)| \sin(\omega t + \theta + \angle T(j\omega))$$


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If  $T(s) = \frac{4}{s+2}$

a) Determine  $T(j\omega)$

b) Determine  $|T(j\omega)|$  & plot

c) Determine  $\angle T(j\omega)$  & plot

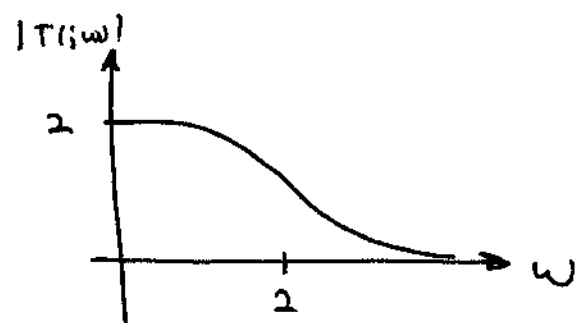
d) Determine sss output if

$$v_i = 9 \sin(2t + 15^\circ)$$

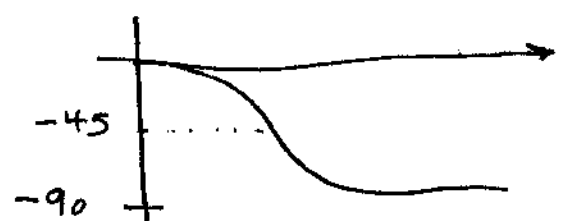

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a)  $T(j\omega) = \frac{4}{j\omega + 2}$

b)  $|T(j\omega)| = \frac{4}{\sqrt{4 + \omega^2}}$



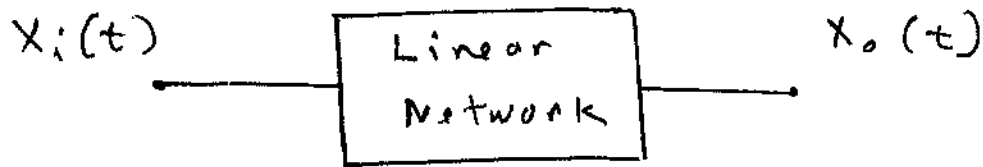
c)  $\angle T(j\omega) = -\tan^{-1} \frac{\omega}{2}$



d)  $V_{oss}(t) = 9 \cdot \frac{4}{2\sqrt{2}} \sin(2t + 15^\circ - 45^\circ)$



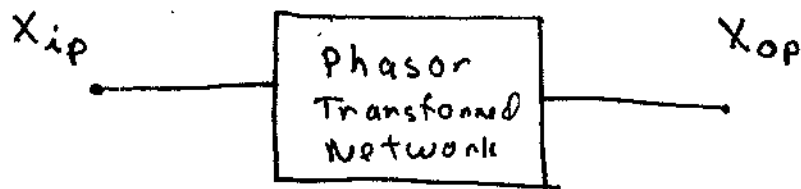
# Relationships Between $T(s)$ , $T_p(j\omega)$ and Laplace Transforms



$$X_i(t) = X_m \sin(\omega t + \theta)$$

## Phasor Analysis

→ Obtain Phasor Transformed Network



$$C \longrightarrow \frac{1}{j\omega C}$$

$$L \longrightarrow j\omega L$$

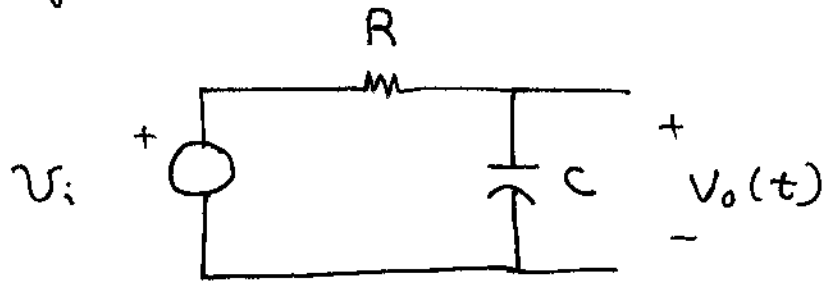
All other elements  $\longrightarrow$  unchanged

Analyze Phasor Transformed Network

$$X_{op} = T_p(j\omega) V_{ip}$$

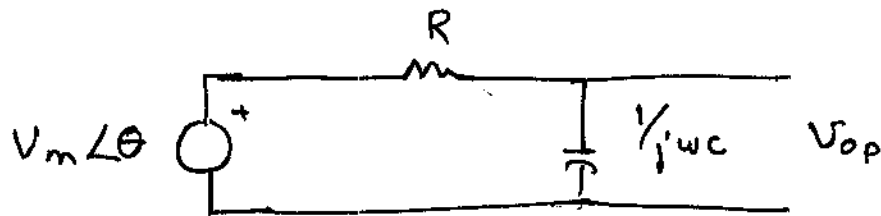
$$T_p(j\omega) = |T_p(j\omega)| \angle \arg(T_p(j\omega))$$

# Example



Obtain  $v_{oss}(t)$  if  $v_i = V_m \sin(\omega t + \theta)$

a) Phasor analysis



$$V_{op} = \left( \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \right) V_m \angle \theta$$

$$V_{op} = \left( \frac{1}{1 + j\omega RC} \right) V_m \angle \theta$$

$$|T_p(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \quad \angle T_p(j\omega) = -\tan^{-1} \omega RC$$

$$v_{oss}(t) = V_m \left[ \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \right] \sin(\omega t + \theta - \tan^{-1} \omega RC)$$

$$X_{ip} = X_m \angle \theta$$

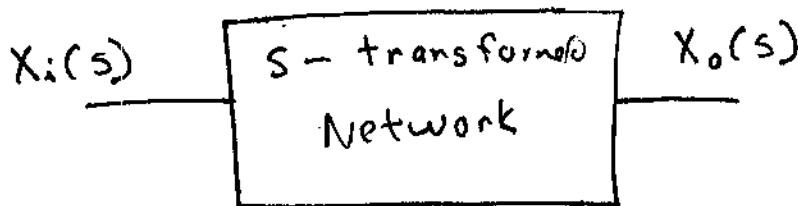
$$\begin{aligned} X_{op} &= T_p(j\omega) V_{ip} \\ &= (|T_p(j\omega)| \angle \arg(T_p(j\omega))) (X_m \angle \theta) \end{aligned}$$

$$X_{op} = (X_m |T_p(j\omega)|) \angle (\theta + \arg T_p(j\omega))$$

$$X_o(t) = X_m |T_p(j\omega)| \sin(\omega t + \theta + \arg T_p(j\omega))$$

## S-domain Analysis

→ Obtain s-transformed network



$$C \rightarrow \frac{1}{sC}$$

$$L \rightarrow sL$$

All other Elements → unchanged.

Analyze s-domain network

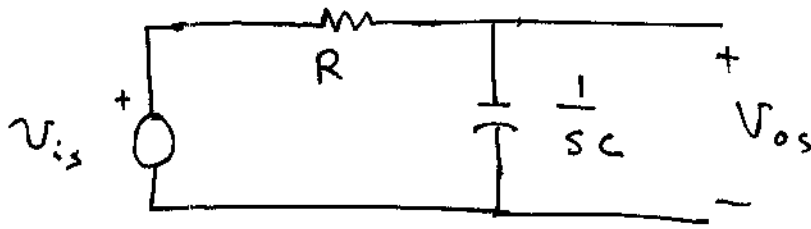
$$X_o(s) = T(s) X_i(s)$$

$$X_o(t) = X_m |T(s)|_{s=j\omega} \sin(\omega t + \theta + \arg(T(s))|_{s=j\omega})$$

Note:

$$T(s) \Big|_{s=j\omega} = T_p(j\omega)$$

## b) S-domain Analysis

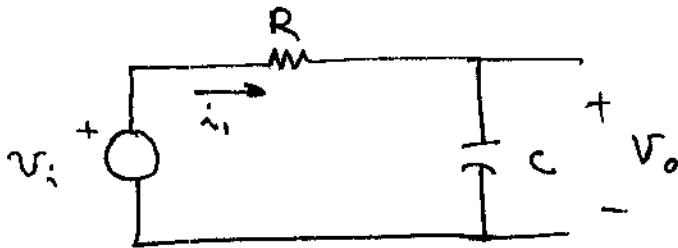


$$V_{os} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_{is}$$

$$\frac{V_{os}}{V_{is}} = T(s) = \frac{1}{1 + RCs}$$

$$V_{oss}(t) = V_m \frac{1}{\sqrt{1 + R^2 c^2 \omega^2}} \sin(\omega t + \theta - \tan^{-1} \omega RC)$$

## c) with differential equations



$$i_1 = \frac{V_i - V_o}{R}$$

$$i_1 = C \frac{dV_o}{dt}$$

$$\left. \begin{array}{l} i_1 = \frac{V_i - V_o}{R} \\ i_1 = C \frac{dV_o}{dt} \end{array} \right\} \frac{V_i - V_o}{R} = C \frac{dV_o}{dt}$$



$$\frac{V_{is} - V_{os}}{R} = CSV_{os}$$

$$V_{os} \left( sC + \frac{1}{R} \right) = \frac{V_{is}}{R}$$

$$V_{os} = V_{is} \left( \frac{R}{1 + RCS} \right)$$

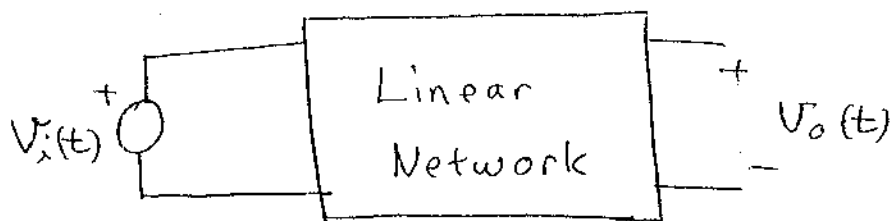
$$V_o(t) = \int^{-1} (V_{is}) \left( \frac{1}{1 + RCS} \right)$$

$$V_o(t) = V_m \left| T_{LP}(s) \right|_{s=j\omega} \sin(\omega t + \theta + \arg(T_{LP}(s))_{s=j\omega})$$

$$V_o(t) = \frac{V_m}{\sqrt{1 + R^2 C^2 \omega^2}} \sin(\omega t + \theta - \tan^{-1} \omega RC)$$

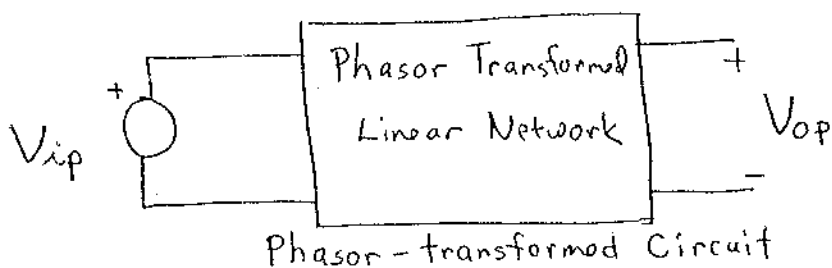
# Sinusoidal Steady State Analysis of Linear Networks - A Review

Consider a linear network with input  $V_i(t)$  and output  $V_o(t)$



$$\text{Assume } V_i(t) = V_m \sin(\omega t + \theta) \quad (1)$$

## Phasor Analysis



Note: •  $V_{ip}$  and  $V_{op}$  are the phasor transforms of  $V_i(t)$  and  $V_o(t)$  respectively

• The phasor transformed linear network is obtained by making the following element transformations

$$\text{— } C \rightarrow \frac{1}{j\omega C} \quad (2)$$

$$\text{— } L \rightarrow j\omega L \quad (3)$$

— All other elements unchanged

Analyzing the phasor-domain network, we obtain the output phasor as the product of the input phasor and a complex function,  $T_p(j\omega)$ , determined by the network. This can be written as

$$V_{op} = T_p(j\omega) \cdot V_{ip} \quad (4)$$

$T_p(j\omega)$  can be written in polar form as

$$T_p(j\omega) = |T_p(j\omega)| \angle \arg(T(j\omega)) \quad (5)$$

where  $\arg(T(j\omega))$  is the angle of the function  $T_p(j\omega)$

The input phasor, from (1), can be written as

$$V_{ip} = V_m \angle \theta \quad (6)$$

Substituting (5) and (6) into (4), we obtain the output phasor

$$V_{op} = (V_m \angle \theta) (|T_p(j\omega)| \angle \arg(T(j\omega))) \quad (7)$$

which can be rewritten in standard polar form as

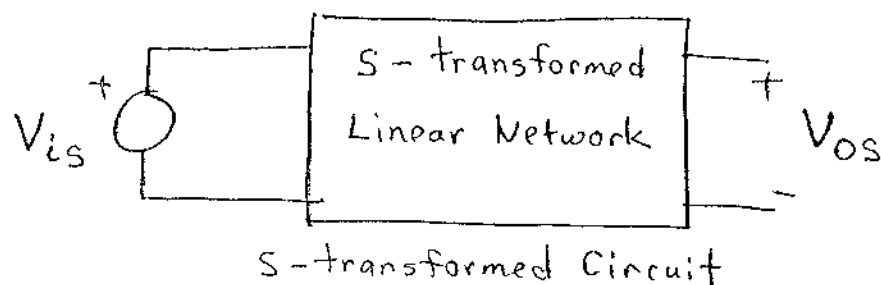
$$V_{op} = [V_m \cdot |T(j\omega)|] \angle (\theta + \arg(T(j\omega))) \quad (8)$$

The sinusoidal steady state output can be obtained from the inverse phasor transform of  $V_{op}$  in (8)

$$V_o(t) = V_m |T_p(j\omega)| \sin(\omega t + \theta + \arg(T(j\omega))) \quad (9)$$

Equation (9) is a key result!

### s-domain analysis



- Note
- $V_{is}$  and  $V_{os}$  are the s-transforms of  $V_i(t)$  and  $V_o(t)$  respectively
  - The s-transformed linear network is obtained by making the following element transformations

$$- \quad C \rightarrow \frac{1}{sC} \quad (2')$$

$$- \quad L \rightarrow sL \quad (3')$$

- All other elements unchanged

Analyzing the s-domain network, we obtain the s-domain output as the product of the s-domain input and an s-domain function,  $T(s)$ , determined by the network. This can be written as

$$V_{os} = T(s) \cdot V_{is} \quad (4')$$

The sinusoidal steady state response can be obtained from the inverse-s transform of  $V_{os}$  which becomes, after transient response terms are neglected,

$$V_o(t) = V_m |T(j\omega)| \sin(\omega t + \theta + \arg(T(j\omega))) \quad (9)$$

where  $T(j\omega) = T(s) \Big|_{s=j\omega}$

Equation (9') is a key result!

Note :  $T(s)$  is called the system transfer function

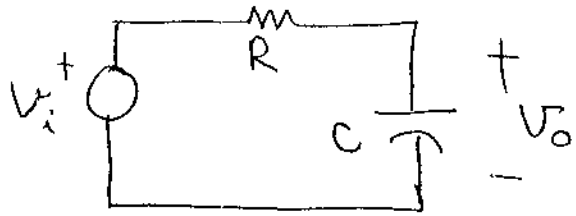
Note :  $T_p(j\omega) = T(s) \Big|_{s=j\omega}$

Note :  $T_p(s) = T(j\omega) \Big|_{\omega = \frac{s}{j}}$

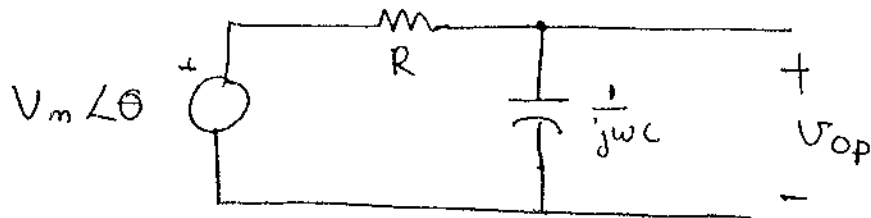
Note :  $T_p(s) = T(s)$

Note : Equations (9) and (9') are identical

Example: Obtain the transfer function and the sinusoidal steady state response of the following circuit using a) phasor analysis ; b) s-domain analysis and c) differential equations. Assume  $V_i = V_m \sin(\omega t + \theta)$



a) Phasor analysis



Phasor-transformed circuit

By voltage divider

$$V_{op} = \left( \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \right) V_m \angle \theta \quad (1)$$

$$V_{op} = \left( \frac{1}{1 + RC\omega j} \right) V_m \angle \theta \quad (2)$$

$$\text{From (2), } T_p(j\omega) = \frac{V_{op}}{V_{ip}} = \frac{V_{op}}{V_m \angle \theta} = \frac{1}{1 + RC\omega j} \quad (3)$$

$$\therefore T(s) = \frac{1}{1 + RCs} \quad (4)$$

Converting  $T_p(j\omega)$  in (2) from rectangular to polar form, we obtain

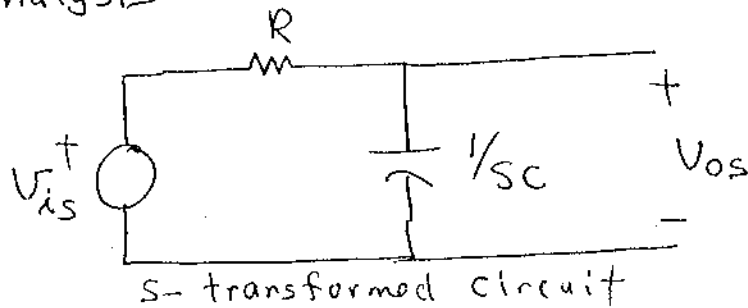
$$V_{op} = \left( \frac{1}{\sqrt{1+R^2C^2\omega^2}} \angle -\tan^{-1}\omega RC \right) V_m \angle \Theta \quad (5)$$

$$V_{op} = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \angle (\Theta - \tan^{-1}\omega RC) \quad (6)$$

Taking the inverse phasor transform of (6) we obtain the sinusoidal steady state response

$$V_o(t) = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \sin(\omega t + \Theta - \tan^{-1}(\omega RC)) \quad (7)$$

b) s-domain Analysis



By voltage divider

$$V_{os} = \left[ \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \right] V_{is} \quad (8)$$

$$V_{os} = \left[ \frac{1}{1+RCs} \right] V_{is} \quad (9)$$

It follows from (9) that the transfer function is

$$T(s) = \frac{1}{1+RCs} \quad (10)$$

From (10),

$$T(j\omega) = \frac{1}{1+RC\omega j} \quad (11)$$

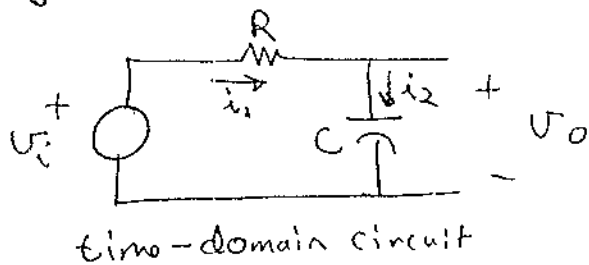
Thus the sinusoidal steady state output is given by

$$V_o(t) = V_m |T(j\omega)| \sin(\omega t + \theta + \angle T(j\omega)) \quad (12)$$

from (12), this becomes

$$V_o(t) = \frac{V_m}{\sqrt{1+R^2C^2\omega^2}} \sin(\omega t + \theta - \tan^{-1}(\omega RC)) \quad (13)$$

c) using differential equations



$$i_1 = \frac{V_i - V_o}{R} \quad (14)$$



$$i_2 = C \frac{dV_0}{dt} \quad (15)$$

equating  $i_1$  and  $i_2$ , we obtain the differential equation

$$\frac{V_i - V_0}{R} = C \frac{dV_0}{dt} \quad (16)$$

I will use Laplace transforms to solve. Taking the Laplace transform of (16)

$$\frac{V_{is} - V_{os}}{R} = C s V_{os} \quad (17)$$

Simplifying (17), we obtain

$$V_{os} = \frac{1}{1 + RCs} \cdot V_{is} \quad (18)$$

If  $V_{is} = V_m \sin(\omega t + \theta)$ , taking the inverse Laplace transform and neglecting the transient response, we obtain

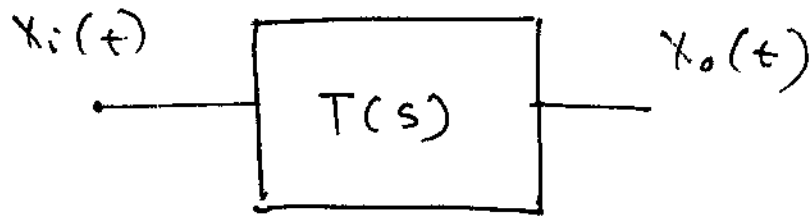
$$V_0(t) = V_m |T(j\omega)| \sin(\omega t + \theta - \arg(T(j\omega))) \quad (19)$$

where  $T(j\omega) = \frac{1}{1 + RCj\omega}$  (20)

Thus,

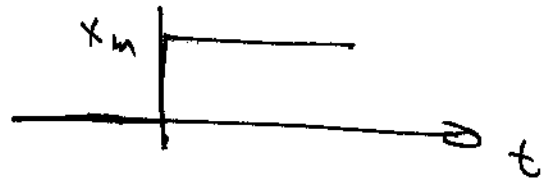
$$V_0(t) = \frac{V_m}{\sqrt{1 + R^2 C^2 \omega^2}} \sin(\omega t + \theta - \tan^{-1} \omega RC) \quad (21)$$

# Step Response of First-Order Networks



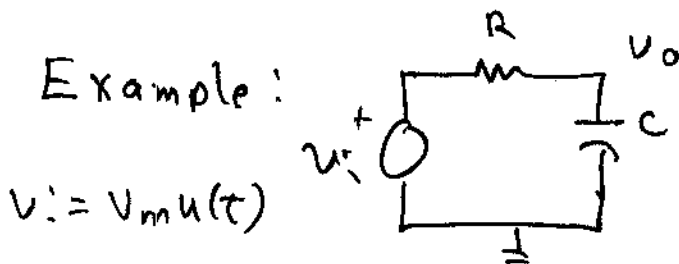
$$T(s) = \frac{N(s)}{S + P}$$

If  $X_i = X_m u(t)$



$$X_o(t) = F + (I - F) e^{-t/\tau}$$

$$\tau = P^{-1}$$



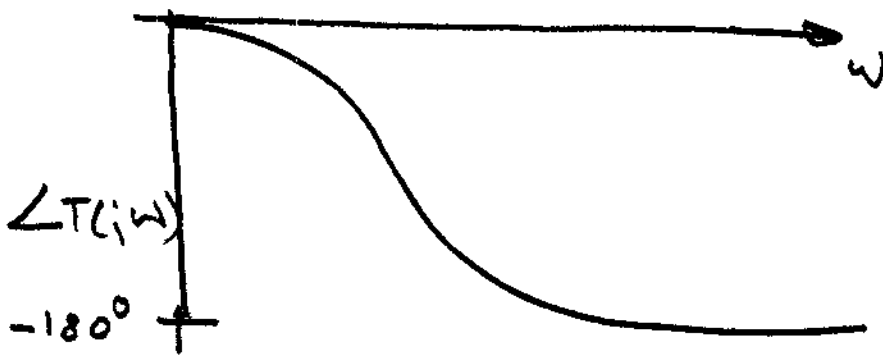
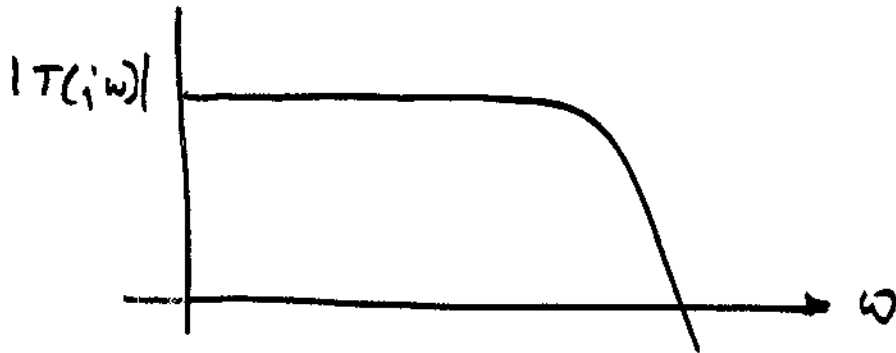
$$\frac{V_o}{V_i} = \frac{1}{1 + RCs} \rightarrow P = \frac{1}{RC}$$

$$V_o(t) = V_m + -V_m e^{-\frac{t}{RC}} = V_m (1 - e^{-t/RC})$$

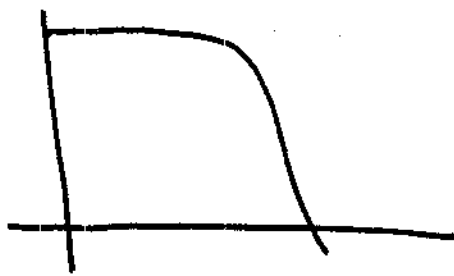
## "Gain" of linear networks

- $T(j\omega)$  represents the gain of a linear network with a sinusoidal input
- $T(j\omega)$  is frequency dependent
- $|T(j\omega)|$  is termed the magnitude of the gain
- $\angle T(j\omega)$  is termed the phase or angle of the gain
- Plots of  $|T(j\omega)|$  and  $\angle(T(j\omega))$  often used to characterize the network
- These plots characterize the "frequency response" of the network

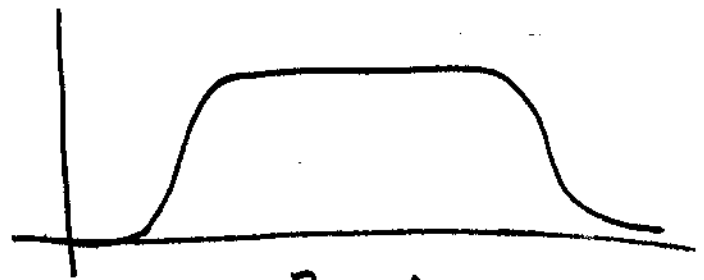
Example :



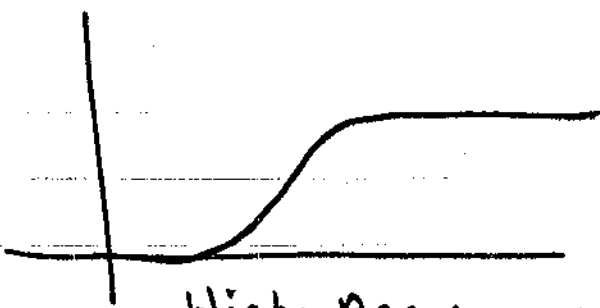
Nomenclature



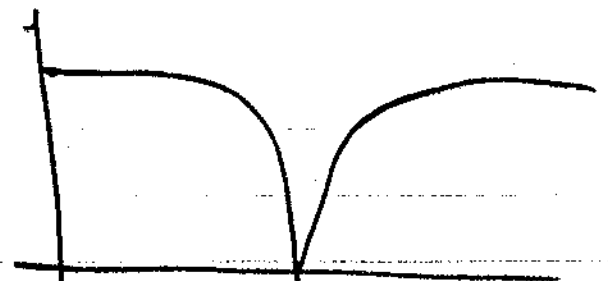
Lowpass



Bandpass



High-pass



Notch